Math 259A Lecture 3 Notes

Daniel Raban

October 2, 2019

1 The Spectral Radius Formula And The Gelfand Transform

1.1 Characters of Banach algebras

Last time, we used the following result to show that morphisms to C^*-algebras are contractive.

Lemma 1.1 (Spectral radius formula). $R(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.

This is really a result about commutative Banach algebras, so to prove it we will discuss the commutative case.

Definition 1.1. Let M be a Banach algebra (with 1_M). A **character** on M is a nonzero linear $\varphi : M \to \mathbb{C}$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in M$. We denote X_M as the space of all characters on M.

Proposition 1.1. *Let M be a Banach algebra. Any $\varphi \in X_M$ is automatically continuous with $\|\varphi\| = 1$.*

Proof. For any $x \in M$, $x - \varphi(x) \cdot 1 \in \ker(\varphi)$. Write $x = (x - \varphi(x) \cdot 1) + \varphi(x) \cdot 1$. Then

$$\|\varphi\| = \sup_{x\in(M)_1} |\varphi(x)| = \sup_{\substack{y\in\ker(\varphi)\\ \lambda\neq 0}} \frac{|\varphi(y + \lambda \cdot 1)|}{\|y + \lambda \cdot 1\|} = \sup_{\substack{y\in\ker(\varphi)\\ \lambda\neq 0}} \frac{1}{\|(y/\lambda) + 1\|}.$$

If $\|y' + 1\| < 1$, then y' is invertible, which means $y' \notin \ker(\varphi)$. So this equals 1. $\square$

Corollary 1.1. *$X_M \subseteq (M^*)_1$ is $\sigma(M^*, M)$-compact (weak* compact).*

Proof. X_M is closed in the weak* topology. $\square$

1.2 The Gelfand transform

Definition 1.2. The **Gelfand transform** $\Gamma : M \to C(X_M)$ is given by $\Gamma(x)(\varphi) := \varphi(x)$.

Proposition 1.2. *The Gelfand transform has the following properties:*

1. *Γ is an algebra morphism.*
2. *$\|\Gamma(x)\|_\infty \leq \|x\|$.*

Theorem 1.1. *If M is a Banach algebra such that any $x \neq 0$ is invertible (a division algebra), then $M = \mathbb{C}$.*

Proof. If $x \in M$, then $\mathrm{Spec}_M(x) \neq \varnothing$, so let $\lambda_x \in \mathrm{Spec}_M(x)$. Then $x - \lambda_x 1$ is not invertible, so $x - \lambda_x = 0$. So $\lambda_x 1 = x$. □

Proposition 1.3. *If M is a Banach algebra and $J \subseteq M$ is a closed, 2-sided ideal, then M/J has a Banach algebra structure given by $\|x + J\| = \inf_{y \in J} \|x + y\|$.*

Proposition 1.4. *If M is a commutative Banach algebra, then there is a correspondence between X_M and the space of maximal, 2-sided ideals of M given by $\varphi \mapsto \ker(\varphi)$.*

Proof. Let $\varphi \in X_M$, and let J be an ideal such that $\ker(\varphi) \subsetneq J$. Let $x \in J \setminus \ker(\varphi)$. Then $x = (x - \varphi(x) \cdot 1) + \varphi(x) \cdot 1$, so 1 is in the span of x and $\ker(\varphi)$, which is contained in J. So J is an ideal containing M and hence equals M. That is, $\ker(\varphi)$ is maximal.

If J is a maximal ideal in M, then $\overline{J}$ is an ideal (using the $\|1 - x\| < 1 \implies x$ is invertible lemma), so J is closed. Then let $\varphi_J : M \to M/J$ be the natural projection map. But since J is maximal, M/J is a division algebra. So $M/J = \mathbb{C}$. This means $J = \ker(\varphi_J)$, where φ_J is a character. □

Proposition 1.5. *If M is a commutative Banach alagebra, then $X_M = \varnothing$ and $x \in M$ is invertible if and only if $\Gamma(x)$ is invertible.*

Proof. If $x \in M$ is invertible, then $\Gamma(x^{-1})$ is the inverse of $\Gamma(x)$. If $x \in M$ is not invertible, then xM is a proper, 2-sided ideal in M. Let $J \subseteq M$ be a maximal 2-sided ideal containing xM. Then $\varphi_J(x) = 0$, so $\Gamma(x)$ is not invertible. □

We can summarize our results in the following theorem.

Theorem 1.2. *Let X be a commutative Banach algebra.*

1. *$X_M \neq \varnothing$.*
2. *Γ is an algebra homomorphism.*
3. *$\|\Gamma(x)\|_\infty \leq \|x\|$ for all $x \in M$.*
4. *$x \in \mathrm{Inv}(M) \iff \Gamma(x) \in \mathrm{Inv}(C(X_M))$.*

1.3 The spectral mapping theorem and the spectral radius formula

Corollary 1.2. *Let M be a commutative Banach algebra, and let $x \in M$. Then* $\mathrm{Spec}_M(x) = \mathrm{Ran}(R(x))) = \mathrm{Spec}_{C(X_M)}(\Gamma(x))$. *Thus,* $R_M(x) = \|\Gamma(x)\|_\infty$.

Proof.

$$\begin{aligned} \lambda \notin \mathrm{Spec}_M(x) &\iff \lambda - x \text{ is invertible in } M \\ &\iff \lambda - \Gamma(x) \text{ is invertible in } C(X_M) \\ &\iff \lambda \notin \mathrm{Range}(\Gamma(x)). \end{aligned}$$

□

Corollary 1.3 (Spectral mapping theorem)**.** *Let M be a Banach algebra, let $x \in M$, and let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with $f(z) = \sum_{n=0}^\infty a_n z^n$. Then* $\mathrm{Spec}_M(f(x)) = f(\mathrm{Spec}(x))$.

Remark 1.1. The function $f(x)$ makes sense, as the sum is absolutely convergent in norm. The radius of convergence is $(\limsup |a_n|^{1/n})^{-1} = \infty)$.

We can now prove the spectral radius formula.

Proof. Let M_0 be the Banach algebra generated by $1, x, f(x), (x-\lambda)^{-1}$ for all $\lambda \in \rho_M(x)$, and $(f(x)-\mu)^{-1}$ for all $\mu \in \rho_M(f(x))$, where ρ denotes the resolvent. Then M_0 is commutative, so $\mathrm{Spec}_{M_0}(x) = \mathrm{Spec}_M(x)$ and $\mathrm{Spec}_{M_0}(f(x)) = \mathrm{Spec}_M(f(x))$. So we may assume that M is commutative.

From the corollary, we have $\mathrm{Spec}_M(x^n) = (\mathrm{Spec}_M(x))^n$ (using the Gelfand transform). So $R_M(x)^n = R_M(x^n) \leq \|x^n\|$. We get that $R_M(x) \leq \liminf_n \|x^n\|^{1/n}$. Let $G(\lambda) = -\lambda \sum_{n=0}^\infty \frac{x^n}{\lambda^n}$. This sum converges absolutely for $|\lambda| > \|x\|$ and converges to $(x-\lambda)^{-1}$. But for $|\lambda| > R_M(x)$ and $\varphi \in M^*$, $\varphi((x-\lambda)^{-1})$ is analytic, and $\lambda \mapsto \varphi(G(\lambda))$ is analytic and agrees with $\varphi((x-\lambda)^{-1})$ there. So we conclude that for every $\varphi \in M^*$, $\lim_{n\to\infty} \varphi(\lambda^{1-n}x^n) = 0$ whenever $|\lambda| > R_M(x)$.

Apply the uniform boundedness principle to $\lambda^{1-n}x^n \in M$, viewed as an element of M^{**}. So there exists $K(\lambda) > 0$ such that $\|\lambda^{1-n}x^n\| \leq K(\lambda)$ for all n. So

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \leq \limsup_{n\to\infty} K(\lambda)^{1/n} |\lambda|^{(n-1)/n} = |\lambda|$$

for each $|\lambda| > R_M(x)$. □

Corollary 1.4. *Let M be a commutative Banach algebra. Then the Gelfand transform $\Gamma : M \to C(X_M)$ is an isometry if and only if $\|x^2\| = \|x\|^2$ for all $x \in M$.*

Proof. Suppose Γ is an isometry. We have $R(x)^2 = R(\Gamma(x))^2 = \|\Gamma(x)\|^2$, and $R(x^2) = R(\Gamma(x^2)) = \|\Gamma(x^2)\|$. These are equal, so $\|x\| = \|\Gamma(x)\| \implies \|x^2\| = \|x\|^2$.

Conversely if $\|x^2\| = \|x\|^2$, then $\|x\| = R(x)$ by the spectral radius formula (we did this argument before). □

1.4 Characterization of commutative C^*-algebras

Recall the Stone-Weierstrass theorem.

Theorem 1.3 (Stone-Weierstrass). *Let X be compact, and let $M \subseteq C(X)$ be a norm-closed, *-closed subalgebra with $1 \in M$ that separates points (i.e. for all $t_1 \neq t_2 \in X$, there is an $f \in M$ such that $f(t_1) \neq f(t_2)$). Then $M = C(X)$.*

Theorem 1.4 (Gelfand). *Let M be a commutative C^*-algebra.*

1. *If $\varphi \in X_M$, then $\varphi = \varphi^*$; i.e. $\varphi(x^*) = \varphi(x)^*$ for all x.*
2. *$\Gamma : M \to C(X)$ is a *-algebra isometric isomorphism.*

Proof. If $x = x^* \in M$, then $\varphi(x) \in \mathrm{Spec}_M(x) \subseteq \mathbb{R}$.

By the first part, $\Gamma(M)$ is *-closed. By definition $\Gamma(M)$ separates points: $\varphi_1 \neq \varphi_2$ means that there is an x usch that $\varphi_1(x) \neq \varphi_2(x)$. By the Stone-Weierstrass, $\overline{\Gamma(M)} = X$. By the C^*-algebra axiom, $\|x^2\| = \|x\|^2$, so Γ is isometric. $\square$

1.5 Continuous functional calculus

Lemma 1.2. *Let M be a commutative C^*-algebra. If $x \in M$ and M is generated by x, then $X_M \simeq \mathrm{Spec}(x)$ via $\varphi \mapsto \varphi(x)$.*

Example 1.1. Let $T \in B(H)$ be normal ($T^*T = TT^*$). Then the spectrum of $C^*(\{T\})$ can be identified with $\mathrm{Spec}(T)$.

So if M is a C^*-algebra, $x \in M$ is normal, and $f \in C(\mathrm{Spec}(x))$, we can think of $f(x) \in M$ by $f(x) = \Gamma^{-1}(f)$. This is **continuous functional calculus**.